Investigations of Linear System Obtained by Element-Free Galerkin Method – Influence of Imposing Method for Boundary Condition –

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1 Introduction

A mesh generation procedure must be needed before using general finite element method (FEM) for solution of the partial differential equations. However, it costs a lot of time to divide the region into a set of finite elements. Thus, the time required for discretizing the partial differential equation or solving the resulting linear system is much shorter than that for the element generation. On the other hand, meshless approaches do not require finite elements of a geometrical structure. The necessary information is only locations of nodes which are scattered in the region and on the boundary. For these reasons, various meshless approaches have been developed, such as the diffuse element method, the Element-Free Galerkin (EFG) method [1] and the Meshless Local Petrov-Galerkin (MLPG) method [2].

The purpose of the present study is to implement the variable preconditioned Minimal Residual method with generalized Jacobi and to solve the linear system obtained by EFG with two types of imposing method for the boundary condition.

2 Element-Free Galerkin Method

In this section, we derive the weak form of the Poisson problem in the analytic two-dimensional region Ω and discretize it by means of the Element-Free Galerkin (EFG) method. For the simplicity, we assume that the region Ω is bounded by essential boundary Γ_{D}. The governing equation is expressed as

$$-\Delta u = f, \tag{1}$$

where $u(x, y)$ denotes unknown function and $f(x, y)$ denotes the given functions. Furthermore, the boundary conditions on Γ_{D} is assumed as $u = \bar{u}$. Here, $\bar{u}$ denotes the given function.

EFG is based on the global Galerkin formulation, and the boundary conditions are satisfied by the Lagrange multiplier. That is to say, the governing equation and boundary conditions can be proved equivalent to the following weak form:

$$\int_\Omega \nabla u \cdot \nabla \delta u \; d\Omega - \int_\Omega p \delta u \; d\Omega + \int_\Gamma \lambda \delta u \; d\ell + \int_\Gamma \delta\lambda (u - \bar{u}) \; d\ell = 0. \tag{2}$$

The function δu and $\delta\lambda$ are arbitrary functions and we call these functions test functions. Besides, u is the trial function and λ denotes the Lagrange multiplier. Here, when the trial and test functions are taken from the same functional space such as $u, \delta u \in \mathrm{span}(\phi_1, \phi_2, \cdots, \phi_M)$, $\lambda, \delta\lambda \in \mathrm{span}(N_1, N_2, \cdots, N_n)$, the weak form is discretized as

$$\begin{bmatrix} W & B \\ B^T & O \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}. \tag{3}$$

where ϕ_j denotes the shape function of EFG for j-th node and N_j denotes the shape function for j-th boundary node. In the present study, the delta function and the first order B-spline is adopted for the functions N_j. Furthermore, the matrices A, B and vectors $\boldsymbol{f}$, $\boldsymbol{g}$ are defined as follows.

$$(W)_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \; d\Omega, \quad (B)_{ij} = \int_\Gamma \phi_i \; N_j \; d\ell,$$
$$(\boldsymbol{f})_i = \int_\Omega \phi_i p \; d\Omega, \quad (\boldsymbol{g})_j = \int_\Gamma N_j \bar{u} \; d\ell.$$

Here $(\;)_{ij}$ represents the (i, j)-th matrix element and $(\;)_i$ represents the i-th vector component.

We can see from (3), the left lower block matrix of the coefficient matrix becomes zero matrix. From this reasons, the stationary iterative method, such as Jacobi method and SOR method can not be applied for the solver.

3 Variable Preconditioned Krylov Subspace Method

It is well known that a preconditioning strategy can improve the performance for solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$ using the Krylov subspace method and various preconditioning strategies have been developed and numerically investigated. Here, A, $\boldsymbol{x}$ and $\boldsymbol{b}$ denote a coefficient matrix, an unknown vector, and a known vector, respectively. Generally, a preconditioned matrix M is determined by incomplete LU decomposition, and a vector $M^{-1}\boldsymbol{r}_k$ is calculated at k-th iteration by using backward substitution or incomplete Cholesky factorization. Here, $\boldsymbol{r}_k$ denotes residual vector at k-th iteration. The calculation time of solving linear system is relatively large for the preconditioned part.

K. Abe *et al.* developed new preconditioning strategy which is called the variable preconditioning method [3]. Variable Preconditioned (VP) GCR has two nested iterations for GCR and variable preconditioning for GCR are called as outer-loop and inner-loop, respectively. In the preconditioned procedure, the residual equation is solved to determine the preconditioner for the outer-loop. The algorithm of VPGCR can be extended for symmetric matrix solver by using other Krylov subspace method. The algorithm of variable preconditioned Minimal Residual (VP-

Let $\boldsymbol{x}_0$ be an initial guess.
$c_1 = -1, c_0 = 1, s_1 = 0, s_0 = 0$
$\boldsymbol{q}_0 = 0, \boldsymbol{v}_0 = 0, \boldsymbol{v}_{-1} = 0$
$\boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{x}_0$
Roughly solve $A\boldsymbol{z}_0 = \boldsymbol{r}_0$
$\beta_0 = \sqrt{(\boldsymbol{r}_0, \boldsymbol{z}_0)}, \eta_1 = \beta_0$
$\boldsymbol{q}_1 = \boldsymbol{r}_0/\beta_0, \boldsymbol{z}_1 = \boldsymbol{z}_0/\beta_0$
for $k = 1, 2, \cdots$, until $|\eta_{k+1}|/\beta_0 \leq \varepsilon$ **do**
$\boldsymbol{r}_k = A\boldsymbol{z}_k - \beta_{k-1}\boldsymbol{q}_{k-1}$
$\alpha_k = (\boldsymbol{r}_k, \boldsymbol{z}_k)$
$\boldsymbol{r}_{k+1} = \boldsymbol{r}_k - \alpha_k \boldsymbol{q}_k$
Roughly solve $A\boldsymbol{z}_{k+1} = \boldsymbol{r}_{k+1}$
$\beta_k = \sqrt{(\boldsymbol{r}_{k+1}, \boldsymbol{z}_{k+1})}$
$\hat{\gamma}_k = -c_k\alpha_k - c_{k-1}s_k\beta_{k-1}$
$\gamma_k = \sqrt{\hat{\gamma}^2 + \beta_k^2}$
$\delta_k = s_k\alpha_k - c_k c_{k-1}\beta_{k-1}$
$\epsilon_k = s_{k-1}\beta_{k-1}, c_{k+1} = \hat{\gamma}_k/\gamma_k, s_{k+1} = \beta_k/\gamma_k$
$\boldsymbol{v}_{k+1} = (\boldsymbol{z}_k - \epsilon_k \boldsymbol{v}_{k-1} - \delta_k \boldsymbol{v}_k)/\gamma_k$
$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + c_{k+1}\eta_k \boldsymbol{v}_{k+1}$
$\boldsymbol{q}_{k+1} = \boldsymbol{r}_{k+1}/\beta_k, \boldsymbol{z}_{k+1} = \boldsymbol{z}_{k+1}/\beta_k$
$\eta_{k+1} = s_{k+1}\eta_k$
end for

Fig. 1. The algorithm of variable preconditioned Minimal Residual (VPMinRes) method.

MinRes) is shown in Fig. 1.

In the inner-loop procedure of VP Krylov subspace method, the residual equation can be solved roughly by using some iterative method with only a few iteration, and a stationary iterative method such as Jacobi method and SOR method is adopted for the solver. As we mentioned above, however, the stationary iterative method can not be applied for the linear system (3). From this reason, the Generalized Jacobi method is adopted for the solver for inner-loop [4].

To adopt Generalized Jacobi method for the inner-loop solver, we transform the coefficient matrix A of the linear system (3) as

$$A = D + L + U = (D + T) + L + (U - T), \qquad (4)$$

where D, L and U denote a diagonal, lower and upper matrix of A. In addition, the elements of the diagonal matrix $T = \mathrm{diag}(t_1, t_2, \ldots, t_n)$ are selected as follows.

$$t_i = \mathrm{sgn}(a_{ii}) \times \max_j |a_{ij}| \qquad (5)$$

$$t_i = \mathrm{sgn}(a_{ii}) \times 1.0 \qquad (6)$$

$$t_i = \mathrm{sgn}(a_{ii}) \times \sum_{i \neq j} |a_{ij}| \qquad (7)$$

Here, a_{ij} denotes a (i, j)-th matrix element of the coefficient matrix A.

4 Numerical Results and Conclusion

As we mentioned above, two types of imposing method for boundary condition are adopted. One is the delta function and the other is first order B-spline for the shape functions N_j. Moreover, VPMinRes with generalized Jacobi is adopted for the solver of the linear system (3), and the residual histories of the system are shown in Fig. 2 and Fig. 3.

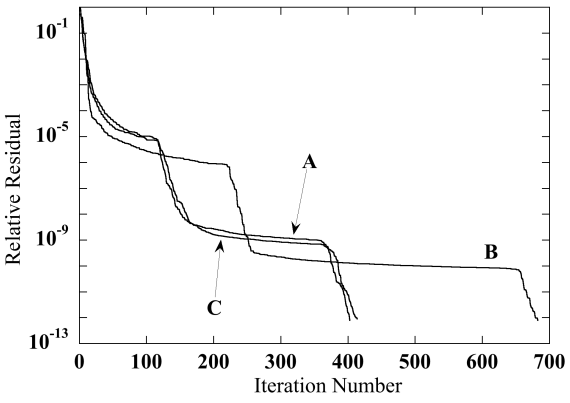

Fig. 2. The residual history of VPMinRes. The boundary conditions are implemented by using the first order B-spline. A: case with (5), B: case with (6), C: case with (7)

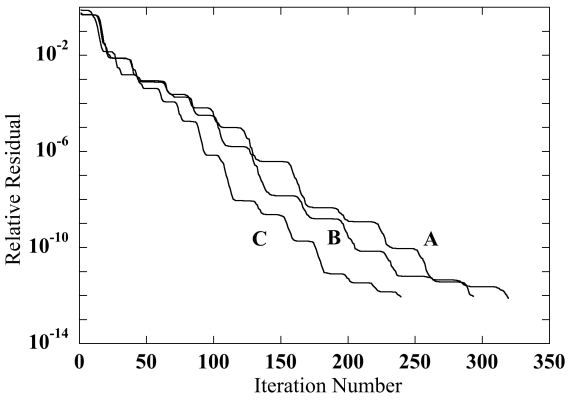

Fig. 3. The residual history of VPMinRes. The boundary conditions are implemented by using the Delta function. A: case with (5), B: case with (6), C: case with (7)

We can see from these figure that if the first order B-spline function is adopted for imposing method, it takes much iteration to solve the system than that of Delta function. This result is derived from the condition number of the coefficient matrix. Actually, the condition number of the coefficient matrix A with first order B-spline and the Delta function are 1.9684×10^{16} and 352.10. The supposable cause is an accuracy of numerical integration of the B-spline.

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