A Multigrid Poisson Solver for Yin-Yang Grid.

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1 Introduction: Solar Dynamo and Simulation Strategy

The sun has a multi-layered structure, consisting of the inner core, the radiative zone, and the convective zone from the center to the surface. Since it is mostly composed of hydrogen and helium gases at the state of plasma, the electric conductivity is high enough to sustain electric currents. The magnetic field is then expected to be generated due to the convective motion of the plasma in the solar interior. We call this process as "solar dynamo" process.

It is well known that the dynamo process also works in the liquid metal which makes up the outer core of the earth. While the earth has quasi-steady, well-organized, dipole magnetic fields, solar magnetic fields show time-varying features due to cyclic sunspot (active region) formations. Though 11-year activity cycle of the sun is believed as a consequence of solar dynamo process, its precise mechanism remains poorly understood in spite of theoretical and observational works for many years.

The compressible MHD equation governs the motion of the plasma in the sun. Since it consists of eight non-linear partial differential equations, it is impossible for us to obtain exact solutions analytically without simplifications. We thus often attack the complex solar dynamo problem by means of computer simulations.

When discretizing the MHD equation, numerical time step is limited by the CFL condition, that is $\Delta t < \Delta x / C_s$, where C_s is the sound speed, Δt is the numerical time step, and Δx is the grid spacing. This is a necessary condition for the stability of explicit finite-difference methods, and states that the numerical domain of dependence must contain the analytical domain of dependence.

Since the gap between the sound and fluid velocities in the sun is very large (i.e., Mach number is $\mathcal{O}(10^{-4})$), the numerical time step Δt is enormously smaller than the typical time scale of the fluid motion [1]. It is thus difficult to carry out the solar dynamo simulation with a practical time step when solving the compressible MHD equation.

To settle this issue associated with the CFL condition, we take a research strategy in which we remove acoustic waves by applying "Low-Mach number approximation" to the MHD equation [2]. The approximated MHD equation is summarized as follows:

$$\begin{aligned} \frac{\partial \boldsymbol{f}}{\partial t} &= -\nabla \cdot (\boldsymbol{v}\boldsymbol{f}) - \nabla \pi + \rho \boldsymbol{g} , \\ \frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} h) &= \nabla \cdot \kappa \nabla T , \\ \nabla \cdot \boldsymbol{v} &= \frac{1}{\rho \frac{\partial p}{\partial \rho}} \left[\frac{1}{\rho c_p} \frac{\partial p}{\partial T} \left(\nabla \cdot \kappa \nabla T \right) \right] \equiv S . \end{aligned}$$

The symbols have their usual meanings, and π is the second order term in the low Mach number expansion of the pressure, that is,

$$p(x,t) = p_0(t) + \mathcal{M} p_1(t) + \mathcal{M}^2 \pi(x,t) .$$

Since the higher order terms of the Mach number $\mathcal{M}$ are neglected, the numerical time step is limited by the fluid velocity in this approximation. Though this enables us to carry out the dynamo simulation with a practical time-step associated with fluid motions, there exists a numerical disadvantage. Namely, we should solve a Poisson equation for maintaining the divergence free of velocity fields.

The implementation of a fast Poisson equation solver to the simulation code is a key for the efficient dynamo simulation with low Mach number approximation. In this paper, we report the development of a fast Poisson equation solver with Multigrid method, and its application to "Yin-Yang grid", which is a type of chimera grid for spherical geometry and has been adopted in our solar dynamo simulation code [4].

2 Multigrid Method and Yin-Yang Grid

The Multigrid method provides algorithms which can be used to accelerate the convergence of basic iterative methods, such as Jacobi , Gauss-Seidel, and SOR methods., by global correction from time to time, accomplished by solving the target equation on coarse grid [3]. The key idea behind multigrid method is to reduce efficiently long wavelength components of residuals with calculations on coarser grids. We adopt this method in the fast Poisson equation solver which will be implemented into our solar dynamo simulation code.

Yin-Yang grid is a kind of overset (Chimera) grid applied to a spherical shell, as shown in Fig.1 [4]. It is recognized that the popular latitude-longitude spherical grid has the "pole problems" that refer to two different kinds of difficulty in numerical calculations; one is the coordinate singularity on the poles, and the other is the grid convergence near the poles. It is a great advantage of the Yin-Yang grid that it has neither a coordinate singularity, nor grid convergence. Since the Yin-Yang grid has been adopted in our dynamo simulation code, we developed the fast Poisson equation solver for the Yin-Yang grid in anticipation of its implementation into the dynamo code.

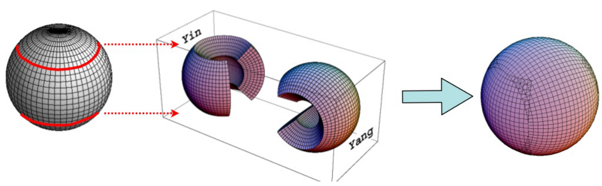

Fig. 1. Yin-Yang Grid which is adopted in our dynamo simulation code [4].

3 Test Problem for Multigrid Poisson Solver

We choose, as the test problem for our Multigrid Poisson solver, the boundary value problem for the magnetic field in the vacuum outside the sphere of $r \geq 1$. When assuming the vacuum as an insulator, the magnetic field should be written by the gradient of a potential field in $r \geq 1$, that is $\boldsymbol{B} = -\nabla\psi$. From the Maxwell equations, we can derive a Laplace equation for the potential field:

$$\nabla^2\psi = 0, \ (r > 1) \ .$$

When applying the coordinate transformation of ζ; $\zeta = 1/r$, the equation can be rewritten, in $0 \leq \zeta \leq 1$, by

$$\left[\zeta^2 \frac{\partial^2}{\partial\zeta^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = 0 \ .$$

The potential problem is now converted into the boundary value problem defined inside a unit sphere of $0 \leq \zeta \leq 1$.

We apply our Multigrid Poisson solver to this problem. The boundary condition of the potential field is given by the radial component of the magnetic filed on the boundary $r = \zeta = 1$ which is given by the analytic formula for the dipole, quadrupole and octapole magnetic field, and sample observational data for the earth and planetary magnetic fields. In our Multigrid poisson solver, the Jacobi method is used as the smoother. The V-cycle is repeated for a couple of times until we get the convergence.

[Convergence Performance]
With applying our Multigrid Poisson solver on the Yin-Yang grid, we obtained the numerical solutions of dipole, quadrupole and octapole magnetic fields as simplest example models. We confirmed that numerical solutions could reproduce the analytic ones, and our Multigrid Poisson solver delivered required performance. Fig.2 demonstrates the convergence times to obtain each numerical solution for the comparison among the numerical schemes adopting Jacobi method with and without multigrid acceleration. This verifies that the convergence performance is improved by Multigrid acceleration: It yields 6–29 times shorter convergence time.

We finally applied our Multigrid Poisson solver to the boundary value problem for the potential fields of the earth. The boundary condition on the earth surface is given by observational data [5]. The magnetic field line we obtained for the earth is visualized in Fig.3.

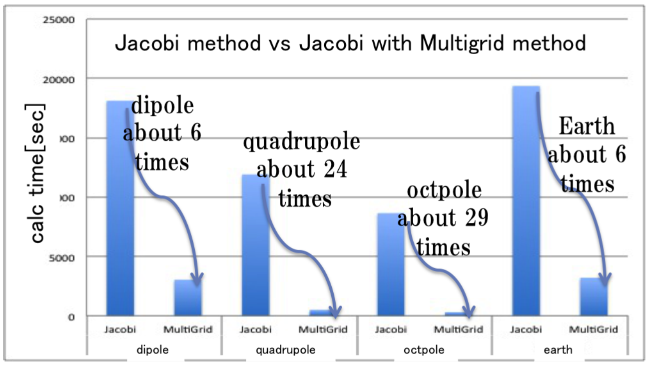

Fig. 2. Convergence performance for different models.

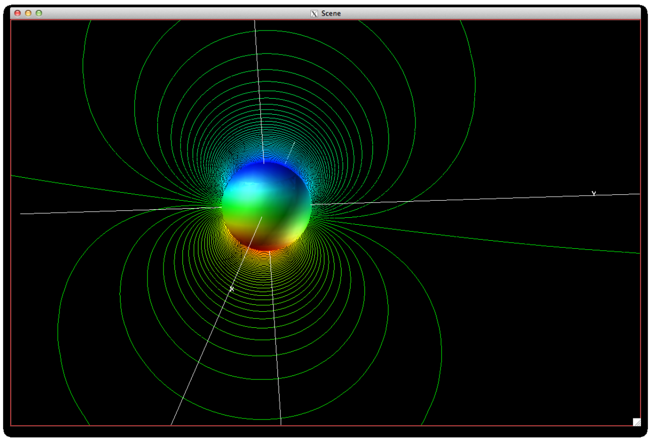

Fig. 3. Visualization of magnetic potential for the Earth

4 Conclusion

The origin of magnetic fields is a longstanding issue in the solar physics and has been explored mainly by numerical simulations. The solar dynamo simulation is, however, suffering from the limitation on the numerical time step due to the CFL condition: Since the gap between the sound and fluid velocities in the sun is very large, the numerical time step is enormously smaller than the typical time scale of the fluid motion.

We thus take a research strategy in which we remove acoustic waves by applying "Low Mach number approximation" to the MHD equation, which enables us to carry out the dynamo simulation with a practical time step associated with fluid motions.

In this paper, we developed the Multigrid Poisson equation solver, and applied it to "Yin-Yang Grid"[4]. This is because it is a key element for efficient dynamo simulation with low Mach number approximation.

Our Multigrid Poisson solver was applied to the boundary value problem for the various type of magnetic fields in the vacuum outside the unit sphere. Then we obtained the numerical solutions of dipole, quadrupole, octpole, and planetary magnetic fields. It was confirmed from the test problems that our Multigrid Poisson solver delivered the required performance and yielded the 6–29 times shorter convergence time. The implementation of Multigrid Poisson solver into our dynamo simulation code is beyond the scope of this paper, but is our future work.

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