Two-Phase Adiabatic Flow Analysis

Using Semi-Lagrangian Galerkin Method

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Abstract

The purpose of this study is an analysis of two-phase flow assuming adiabatic state using the semi-Lagrangian Galerkin method. The two-phase flow means that flow field consists of two fluids which have different densities. In this study, conservations of mass and momentum of adiabatic flows are employed as the governing equations. A fluid is assumed as liquid. Therefore, the Birch-Murnaghan equation of state is used to the energy equation. The semi-Lagrangian Galerkin method is a numerical technique which combines the semi-Lagrange method and implicit method. The governing equations are divided into two parts: the advection and non-advection terms. The characteristic method is applied to the advection term. Both advection and non-advection terms are discretized by the Galerkin method in the spatial direction. The Hermite interpolation function which is based on the complete third order polynomials interpolation using triangular finite element is used for density and velocity in advection and non-advection calculations. Two-phase flow of different densities is calculated.

Keywords – finite element method, adiabatic flows, semi-Lagrange Galerkin method, two-phase flow, Hermite interpolation function, characteristic method

1 Introduction

The adiabatic flows mean the compressible flows assuming adiabatic state in the present study. In actual natural phenomena, almost flows are compressible flows. However, actually, the incompressibility assumption is often used to analyse compressible flows. Numerical results by the analysis assuming incompressible flows are almost the same as those of the compressible flows, especially at low at low velocity flows. Analysis of the compressible flows can be conducted depending on whether velocity is high or not. The numerical results of adiabatic flows are almost the same as those of compressible flows in case of moderate velocity flows.

In the governing equation of the flow problems, the advection term and the diffusion term are included. In case of either term is superior, the characteristic of flows are different. If the advection term is superior, the computation has an inclination to be unstable. Therefore, depending on the characteristic of flows, the suitable appropriate technique is required. For preventing this instability, the characteristic method is used in this study. The terms of temporal differentiation and advection are expressed in the form of material differentiation and transformed by the characteristic method. In addition, the advection calculation is forwarded by the non-advection calculation. After calculation of the advection term by the semi-Lagrangian method, the non-advection term is calculated by the implicit method. This technique is called the semi-Lagrangian Galerkin method. In the advection and non-advection calculation, the Hermite interpolation function is used for velocity and density. The Hermite interpolation function is composed of 10 degrees of freedom, i.e., function values at the three nodes, values of the first derivative, and a function value at the center of gravity. Therefore, the Hermite interpolation function is the complete third order polynomial interpolation function on the triangular element.

2 Governing Equations

As the governing equation, the following equations are employed: conservation of mass,

\[
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho u \cdot \nabla u = 0
\]

Conservation of momentum,

\[
\rho \left( \frac{\partial u_i}{\partial t} + u \cdot \nabla u_i \right) - \tau_i = 0,
\]

\[
\tau_i = -p \delta_{ij} + \lambda \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

And state equation. The Birch-Murnaghan equation of state is used assuming liquid;

\[
p = \frac{3}{2} \lambda \left( \frac{\rho}{\rho_0} \right)^3 - \frac{5}{2} \left( \frac{\rho}{\rho_0} \right)^5
\]

Coefficient of bulk viscosity is expressed as:

\[
\lambda = -\frac{2}{3} \mu
\]

3 Numerical Method

3.1 Definition of Derivative

The position of a fluid particle at time \( \tau \) which was at position \( x \) at time \( t \) is denoted by \( X(x, t; \tau) \). The trajectory of particle at time \( \tau \) is shown by the following ordinary differential equations.

\[
\frac{dX}{d\tau} = u(X(x, t; \tau)), \quad X(x, t; t) = x
\]

The relationship of the time derivative can be expressed as;

\[
\frac{du}{d\tau} = \frac{\partial u}{\partial \tau} + u \cdot \nabla u = \frac{d}{d\tau} \left| \frac{dX}{d\tau} \right|_{\tau = \tau} = \frac{du}{dt}
\]

3.2 Semi-Lagrangian Galerkin method

The governing equations can be divided into two parts; the advection and non-advection terms. For the advection term, the semi-Lagrangian method is applied.

\[
\rho_{n+1} = \rho + \Delta t \rho_{n+1}^a
\]

\[
u_{n+1} = \nu + \Delta t \nu_{n+1}^a
\]

where

\[
\rho_{n+1} = p(X(x, t; t+1)) \quad \text{at} \quad t = t_n
\]

\[
u_{n+1} = u(X(x, t; t+1)) \quad \text{at} \quad t = t_n
\]

The calculation of the upstream point is expressed as follows:
\[ Z_i^r = x_i - u_i^r \Delta T \]
\[ \Delta X^r_i = \frac{1}{2} \left[ \rho_i \left( Z_i^r + u_i^{r+1} \right) + \rho_i \left( Z_i^r - u_i^{r+1} \right) \right] \]

The positions of upstream points of node and center of gravity are \( X_i^r \) and \( X_i^c \). The elements in which the upstream point belongs are expressed by \( K(X_i^c) \) and \( K(X_i^r) \), respectively. The function values are updated by the advection calculation at nodes and at center of gravity.

### 4 Interface Condition

#### 4.1 Velocity of Interface

On the surface which is the boundary of the two flows, the following equations are valid:
\[
\begin{align*}
[\rho_i (u_i - v_j)] &= 0 \\
[\mu_i (u_i - v_j)] &= \alpha_i n_i
\end{align*}
\]
where
\[ [A] = A^r - A^c \]

The velocity of interface is expressed as follows;
\[
v_i = \frac{\rho_i (u_i - v_j) - \rho_j (u_j - v_j) - \alpha_i n_i}{\rho_i (u_i - v_j) - \rho_j (u_j - v_j)}
\]

#### 4.2 Curvature

For computation of the curvature of an interface, a third order B-spline function is used. The curvature is computed by B-spline, and written as follows;
\[
\kappa = \frac{x \ddot{y} - \ddot{x} y}{(\dot{x}^2 + \dot{y}^2)^\frac{3}{2}}
\]
where
\[
\dot{x} = \frac{dx}{ds}, \quad \ddot{x} = \frac{d^2x}{ds^2}
\]

### 5 Finite Element Equation

The finite element equation of the conservation of mass is expressed as:
\[
\frac{1}{\Delta t} M_{n \rho_i \rho_i^{M^{-1}}} \rho_i \dot{G}_{n \rho_i} U_{n \rho_i^{M^{-1}}} = \frac{1}{\Delta t} M_{n \rho_i \rho_i^{M^{-1}}} (\rho_i^+ - \rho_i^-) \Omega_{n \rho_i}
\]

Similarly, the finite element equation of the conservation of momentum is expressed as;
\[
\frac{\rho}{\Delta t} M_{n \rho_i \rho_i^{M^{-1}}} \rho_i \dot{G}_{n \rho_i} U_{n \rho_i^{M^{-1}}} \delta = \frac{\rho}{\Delta t} M_{n \rho_i \rho_i^{M^{-1}}} (\rho_i^+ - \rho_i^-) \Omega_{n \rho_i}
\]
\[
+ \mu D_{n \rho_i \rho_i^{M^{-1}}} \delta + \mu D_{n \rho_i \rho_i^{M^{-1}}} \delta = \frac{\rho}{\Delta t} M_{n \rho_i \rho_i^{M^{-1}}} (\rho_i^+ - \rho_i^-) \Omega_{n \rho_i}
\]
where coefficient matrices are expressed as follows;
\[
M_{n \rho_i \rho_i^{M^{-1}}} = \int_{\Omega_i} H_{n \rho_i} \rho_i \delta \, d\Omega_i \\
G_{n \rho_i} = \int_{\Omega_i} H_{n \rho_i} \rho_i \delta \, d\Omega_i \\
D_{n \rho_i \rho_i^{M^{-1}}} = \int_{\Omega_i} H_{n \rho_i} \rho_i \delta \, d\Omega_i \\
D_{n \rho_i \rho_i^{M^{-1}}} = \int_{\Omega_i} H_{n \rho_i} \rho_i \delta \, d\Omega_i
\]

### 6 Numerical Examples

In Tokyo bay, the two-phase flow is analysed using the semi-Lagrangian method. A numerical analysis of two-phase flow with low density difference is in case 1. The computational domain is divided into high density area and low density area. A computational domain and boundary conditions are expressed in Figures 1 and 2. On high density area, density is 1.05. On low density area, density is 1.0.

### References
