# Simulation Analysis of the Optimal Vaccination Strategy in the Stochastic Infectious Model

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## 1 Introduction

A prevention of prevalence of infectious diseases is one of important problems in epidemiology. In the past, the public health system has prepared for some strategies such as antibiotics and vaccines to control the infectious disease development. In order to build up more effective strategies, we need a precise theoretical analysis of infectious diseases. From a theoretical viewpoint, the mathematical model which describes the spread of the infectious disease has a very important role. In this paper, we study the stochastic modeling of the infectious disease in populations consisting of four populations: susceptible, infected, recovered and vaccinated ones. First, we propose the stochastic infectious model with vaccination in Section 2. In Section 3, we construct the optimal vaccination strategy for the stochastic infectious model using the stochastic maximum principle and the 4-step scheme. Finally, we show the efficiency of the optimal vaccination strategy by the numerical simulations in Section 4.

## 2 Stochastic Infectious Model with Vaccination

Denoting the ratio of the population size of susceptible (individuals susceptible to the disease), infected, recovered and vaccinated to the total population at time $t$ be $S(t), I(t), R(t)$ and $V(t)$, consider the interaction between each population as shown in Fig. 1.

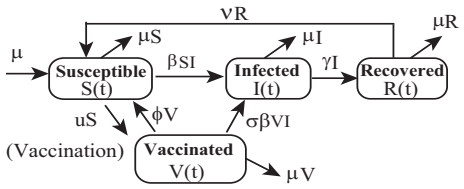


Fig. 1. Interaction between each population [1]

In Fig. 1, $\beta$ presents the transmission rate, $\mu$ the death rate(= the birth rate), $\gamma$ the recovery rate, $\nu$ the rate of loss of immunity, $u$ the vaccination rate, $\phi$ the rate of vaccination waning. A vaccination efficacy is denoted by $1-\sigma$ with $\sigma \in [0,1]$.

Noting that the real infectious disease contains some kinds of random fluctuations caused by changes in the environment, we consider the rate of loss of immunity with a random fluctuation. We replace the rate of loss of immunity $\nu$ by

$$\nu \longrightarrow \nu + \varepsilon\eta(t), \tag{1}$$

where the parameter $\varepsilon$ is a constant and $\eta(t)$ denotes the Gaussian white noise with zero mean and unit covariance.

Using the relation between the Gaussian white noise $\eta(t)$ and the Wiener process $w(t)$ such that $dw(t) = \eta(t)dt$, we have the stochastic infectious model with vaccination:

$$dS(t) = \{\mu - (\mu + u(t))S(t) - \beta S(t)I(t) + \phi V(t) + \nu R(t)\}dt + \varepsilon R(t)dw(t), \tag{2}$$

$$dI(t) = \{\beta S(t)I(t) + \sigma\beta V(t)I(t) - (\mu+\gamma)I(t)\}dt, \tag{3}$$

$$dR(t) = \{\gamma I(t) - (\mu+\nu)R(t)\}dt - \varepsilon R(t)dw(t), \tag{4}$$

$$dV(t) = \{u(t)S(t) - (\mu+\phi)V(t) - \sigma\beta V(t)I(t)\}dt, \tag{5}$$

with the initial conditions

$$S(0) = S_0,\ I(0) = I_0,\ R(0) = R_0,\ V(0) = V_0. \tag{6}$$

Since the death occurs with the same rate $\mu$ as the birth rate, we have

$$S(t) + I(t) + R(t) + V(t) = 1. \tag{7}$$

## 3 Stochastic Optimal Vaccination Strategy

First, we consider the cost function $J(u)$ such that

$$J(u) = E\left\{\int_0^T \ell(x(t), u(t))dt\right\}, \tag{8}$$

where $\ell : R^3 \times R^1 \to R^1$ and is assumed to be continuously differentiable.

Then, we study the optimal control problem of finding the optimal vaccination rate $u^* \in \mathcal{U}$ such that

$$J(u^*) \leq J(u), {}^{\forall} u \in \mathcal{U}, \tag{9}$$

where $\mathcal{U}$ is an admissible control set defined by

$$\mathcal{U} \equiv \{u | 0 \leq u(t) \leq C_p, {}^{\forall} t \in \Theta \equiv (0,T)\}, \tag{10}$$

and where $C_p \in (0\ 1]$ is a constant.

For simplicity of descriptions, we define the vector:

$$x(t) = [x_1(t)\ x_2(t)\ x_3(t)]' \equiv [S(t)\ I(t)\ R(t)]'. \tag{11}$$

We introduce the Hamiltonian $H$ in such a way that

$$H(x,u,p,q) = \langle f(x,u), p\rangle - \ell(x,u) + \langle g(x), q\rangle, \tag{12}$$

where $\langle\cdot,\cdot\rangle$ denotes a Euclidean inner product, $p$ and $q$ are adjoint vectors, $f$ and $g$ are vectors with components:

$$f_1 = \mu - \mu x_1 - \beta x_1 x_2 - u x_1 + \phi v + \nu x_3, \tag{13}$$

$$f_2 = \beta x_1 x_2 + \sigma\beta v x_2 - (\mu+\gamma)x_2,\ f_3 = \gamma x_2 - (\mu+\nu)x_3, \tag{14}$$

$$g_1 = \varepsilon x_3,\ g_2 = 0,\ g_3 = -\varepsilon x_3. \tag{15}$$

and where $v = (1 - \sum_{j=1}^3 x_j)$.

Then, the stochastic maximum principle [2] yields that

$$dx^* = H(x^*, u^*, p, q)_p dt + g(x^*)dw(t),\ x^*(0) = x_0, \quad (16)$$

$$dp = -H(x^*, u^*, p, q)_x dt + q(t)dw(t), \quad p(T) = 0. \quad (17)$$

$$H(x^*, u^*, p, q) = \max_{u \in \mathcal{U}} H(x^*, u, p, q), \quad (18)$$

where $x^*(t)$ is an optimal trajectory of $x(t)$.

We solve the forward-backward stochastic differential equations (16) and (17) using the four-step scheme [3].

Assume that $p(t)$ and $x(t)$ are related by

$$p(t) = \theta(t, x^*(t)), \quad (19)$$

where $\theta$ is some vector-valued function with components $\theta^i(t, x^*(t))$ $(i = 1, 2, 3)$ to be determined.

In the sequel, the asterisks of $x(t)$ and $u(t)$ are omitted for simplicity of descriptions.

Using the Itô's lemma to $\theta^i(t, x(t))$, we have for $i = 1, 2, 3$,

$$d\theta^i = \{\theta^i_t + \left\langle \theta^i_x, f \right\rangle + \frac{1}{2}\mathrm{tr}[(\theta^i_x)'_x gg']\}dt + \left\langle \theta^i_x, g \right\rangle dw. \quad (20)$$

Noting that $p(t) = \theta(t, x(t))$ and $u$ is a function of $p, q$ and $x$, it follows from Eqs. (17) and (20) that

$$\theta^i_t + \left\langle \theta^i_x, f \right\rangle + \frac{1}{2}\mathrm{tr}[(\theta^i_x)'_x gg'] + H(x, \theta, q)_{x_i} = 0. \quad (21)$$

$$q_i(t) = \left\langle \theta^i_x, g \right\rangle, \quad (22)$$

with the terminal condition

$$\theta^i(T, x) = 0, \quad (i = 1, 2, 3). \quad (23)$$

Since Eq. (21) is a deterministic partial differential equation, we can solve Eq. (21) with Eq. (22) under the terminal condition (23).

## 4 Simulations

For simplicity of numerical calculations, we assume that the vaccination has a complete efficiency and recovered does not become reinfected and the vaccinated is included in recovered, i.e., $\nu = \sigma = \phi = 0$. We consider the random fluctuation in the infection rate $\beta$. Setting as $\beta \to \beta + \varepsilon\eta(t)$, we have

$$dS = \{\mu - (\mu + u)S - \beta SI\}dt - \varepsilon SIdw(t), \quad (24)$$

$$dI = \{\beta SI - (\mu + \gamma)I\}dt + \varepsilon SIdw(t), \quad (25)$$

$$dR = \{\gamma I + uS - \mu R\}dt. \quad (26)$$

Numerical simulations are performed under the parameter values: $S_0 = 0.60, I_0 = 0.20, \beta = 0.0024, \mu = 1/73.5, T = 30, \gamma = 1/6570, \varepsilon = 0.05, m = 2.0, n = 2.0, r = 0.8, C_p = 0.7$.

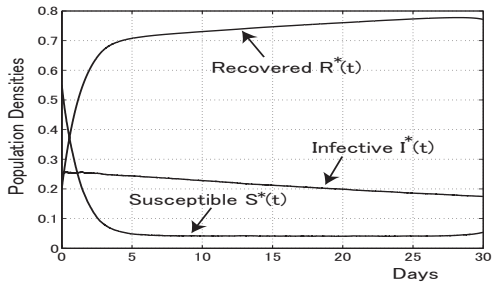


Fig. 2. Optimal Trajectory of Each Population Density

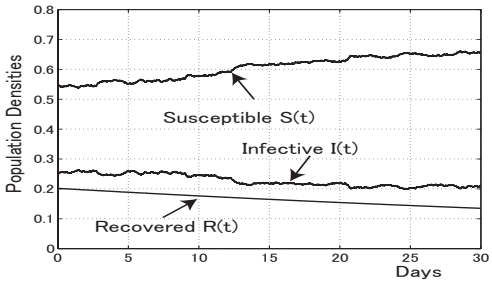


Fig. 3. Trajectory of Each Population Density under No Vaccination

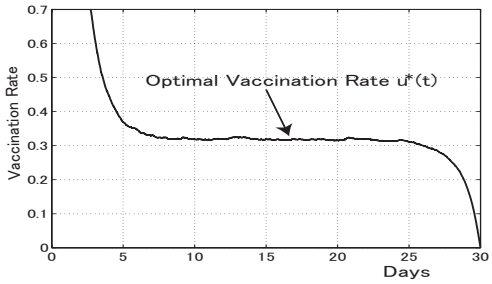


Fig. 4. Time Evolution of Optimal Vaccination Rate

Figures 2 and 3 show the trajectory of each population density under the optimal vaccination strategy and no vaccination. Figure 4 is the trajectory of the optimal vaccination rate $u^*(t)$. From Figs. 2 and 3, by the optimal vaccination strategy, we can see that the infected and the susceptible are decreasing and the recovered is increasing. From Fig. 4, we can know at what rate we will vaccinate. Comparing Fig. 2 with Fig. 3, trajectories under no vaccination have been more significantly impacted by the noise than ones under vaccination because the noise is proportional to $S(t)I(t)$ as shown in Eqs. (24) and (25).

## 5 Conclusions

In this paper, we have considered the stochastic modeling of the infectious disease spreading under vaccination and have studied the stochastic optimal vaccination problem. In the stochastic optimal vaccination problem, we derived the feasible optimal vaccination strategy using the stochastic maximum principle and the four-step scheme. In numerical simulations, for the restriction of numerical calculations, we have considered the optimal control problem of the stochastic SIR model with vaccination. Results of numerical simulations have shown that our optimal vaccination strategy is effective. From simulation results obtained in this paper, we can know how much vaccination coverage is required in order to control the infectious disease with restricted vaccination rate.

## References


[1] J. Arino et al., "Global Results for an Epidemic Model with Vaccination that Exhibits Backward Bifurcation", *SIAM J. Appl. Math.*, 64, 260-276, 2003.

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[3] J. Ma and J. Yong, "Forward-backward Stochastic Differential Equations and their Applications",, Lecture Notes in Mathematics, 1702, Springer-Verlag, 1999.