Equilibrium state for two-dimensional point vortex system with multiple temperatures

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1 Introduction

A concept of a negative temperature state for two-dimensional (2D) point vortex system was first introduced by Onsager to explain large-scale structure formation, for example, Jupiter's Great Red Spot, Naruto eddy, and so on [1]. If the system temperature is negative, a probability proportional to $\exp(-\beta E)$ increases as the system energy increases, so that higher energy configuration is possible, where β is inverse temperature and E is energy.

Motivated in part by his conjecture, we have tackled this problem both numerically and theoretically. In this paper, we will present recent results revealing that an equilibrium state at negative absolute temperature consists of multiple temperature subsystems.

2 Point vortex system

We consider a system consisting of $N/2$ positive and $N/2$ negative point vortices confined in a circular area with radius R. The word "positive vortex" means that the circulation of the vortex is positive. The value of the i-th point vortex Ω_i is either Ω_0 or $-\Omega_0$ where Ω_0 is a positive constant. System energy H is given by

$$H = \frac{1}{2}\sum_i \Omega_i \psi_i \tag{1}$$

$$\begin{aligned} \psi_i = & -\frac{1}{2\pi}\sum_{j\neq i}\Omega_j \log|\boldsymbol{r}_i - \boldsymbol{r}_j| \\ & +\frac{1}{2\pi}\sum_j \Omega_j \log|\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j| \\ & -\frac{1}{2\pi}\sum_j \Omega_j \log\frac{R}{|\boldsymbol{r}_j|} \end{aligned} \tag{2}$$

where $\boldsymbol{r}_i$ is the position vector of the i-th point vortex and ψ_i is the stream function at $\boldsymbol{r}_i$ which corresponds the energy possessed by the i-th point vortex. The effect of the circular wall is introduced by the image vortex located at $\bar{\boldsymbol{r}}_i = R^2\boldsymbol{r}_i/|\boldsymbol{r}_i|^2$. Motions of the vortices are traced by the following equations of motion:

$$\Omega_i\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \Omega_i\frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{3}$$

3 Negative absolute temperature

Here we briefly summarize the concept of the negative absolute temperature introduced by Onsager [1]. Statistical definition of temperature T is given by

$$\frac{1}{T} = \frac{\partial S}{\partial E} = k_B\frac{\partial \log W}{\partial E} \tag{4}$$

where S is entropy, W is density of state and Boltzmann's relation is assumed. The density of state increases as energy increases in normal systems, so that temperature is always positive. On the other hand, if the total phase space volume (total number of states available) is limited, the density of state has a peak at some energy E_0 and decreases $E > E_0$. Thus, T becomes negative at $E > E_0$. The equations of motion of the point vortices are given by Eq. (3). The configuration space coincides with the phase space for the point vortex system. As the total phase space volume is $(\pi R^2)^N < \infty$, Onsager anticipated the existence of the negative temperature state for the point vortex system.

4 Simulation results

To reveal the characteristics of the point vortex system in negative temperature state, we have carried out simulation research.

4.1 Equilibrium distribution

Equilibrium distributions in both negative and positive temperatures are obtained time asymptotically. They are shown in Fig. 1. For the positive temperature case, both

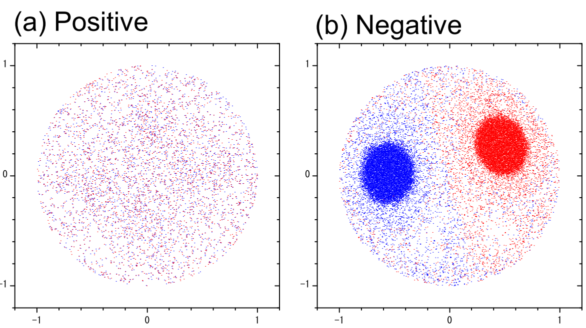

Fig. 1. Typical equilibrium distributions of point vortices at (a) positive and (b) negative temperatures.

sign vortices spread uniformly over the circular area. On the other hand, for the negative temperature case, same sign vortices tend to cluster and form clumps exclusively consisting of single sign.

4.2 Population of point vortices categorized by energy

The energy possessed by each point vortex is defined by ψ_i as shown in Eq. (2). Populations (histograms) as a function of ψ_i are plotted in Fig. 2. For the positive tem-

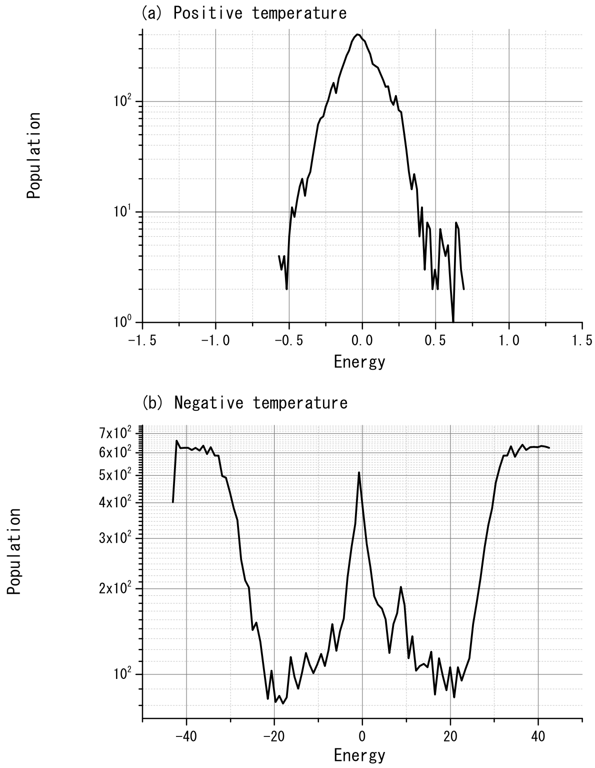

Fig. 2. Population is plotted as a function of energy possessed by each point vortex ψ_i.

perature case, the peak is located at $\psi_i \approx 0$. This is due to the uniform distribution of the point vortices. For the negative temperature case, there are three peaks. The leftmost and the rightmost peaks correspond the clump distributions. The center peak corresponds the background uniform distribution outside the clumps.

To confirm the origin of the peaks located at the both ends, population is recalculated separately for the point vortices inside the clump and the others. The result is shown in Fig. 3. Red line indicates the population of the vortices inside the clumps, and blue line outside the clumps. We categorize the energy regions in 8 parts in Fig. 3. In region (1), the population decreases linearly, which corresponds the distribution is proportional to $\exp(-\beta\Omega_i\psi_i)$. The slope is almost kept constant among the various simulations with $N\Omega_0$ = constant. The center peak resembles the positive temperature case shown in Fig. 2 (a). Indeed, the blue line originates the background vortices located outside the clumps. These observations indicate that the equilibrium distribution consists of the subsystems of different temperature including positive and negative ones.

Another evidence is given by Fig. 4. Equilibrium vortex distribution is rearranged in conjunction with the 8 energy ranges shown in Fig. 3. This figure clearly indicates that the vortices in the cores of the clumps yield the leftmost and rightmost populations. It is surprising that the boundaries of the different temperature vortices are parallel to

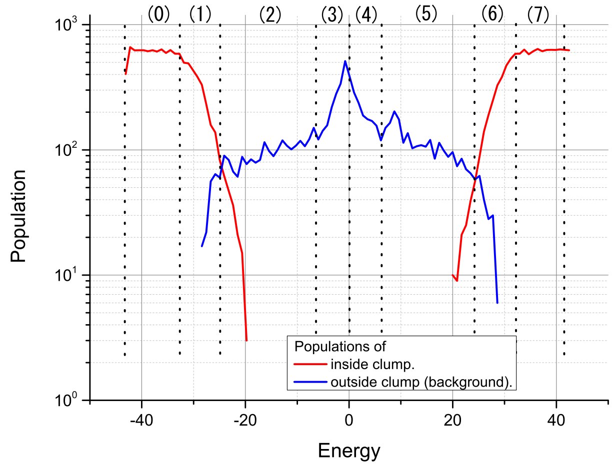

Fig. 3. Energy population is plotted separately for the vortices outside the clumps and inside the clumps

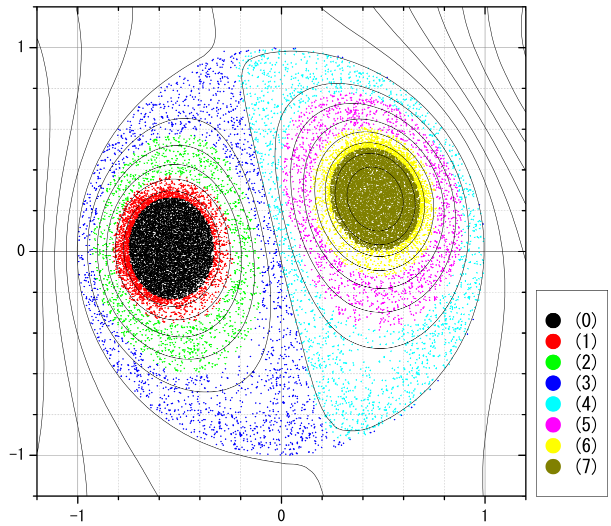

Fig. 4. Equilibrium distribution categorized by 8 energy ranges shown in Fig. 3 is shown. Black line indicates the stream function.

the stream function. As there is no flow (particle motion) across the boundaries, the difference of the temperature is preserved in the equilibrium distribution.

5 Conclusion

In this paper, we have shown that the equilibrium state of 2D point vortex system in negative temperature consists of multiple subsystems of different temperature. The difference of the temperature is maintained due to the no particle transport across the boundary.

Acknowledgement

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References

[1] L. Onsager, "Statistical Hydrodynamics", Nuovo Cimento Suppl., **6**, 279, 1949.